Fluctuations with Time of Scattered-Particle Intensities*

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The fluctuations in counting rate of a particle detector are studied. These may be used to study the coherence properties of the beam. For the case of electromagnetic radiation they may be used to study spectral line shapes. The fluctuations in intensity of scattered particles provide a means of studying fluctuation phenomena in the target.

I. INTRODUCTION

In a recent paper[1] the correlation in counting rates was studied for two detectors counting scattered particles. There it was shown that by such an observation the phase of a scattering amplitude can be measured. In the present paper, we wish to study the correlation of fluctuations in the counting rate of a single detector and of a pair of detectors.

We have in mind a typical scattering experiment, as is illustrated in Fig. 1. A beam of particles is directed on a target. Particles scattered into the detector are counted and recorded. For simplicity of discussion, we shall suppose that the detector output is in the form of an instantaneous electric voltage $C(t)$, at time t, across two terminals T_D. We shall also suppose the detector to have been calibrated so that

$$N_c \equiv \int_0^T C(t)dt \tag{1.1}$$

is the total number of particles entering the detector in a time interval T. The normalization (1.1) suggests that we may call $C(t)$ the *instantaneous counting rate* of the detector.

In any given experiment the counting rate $C(t)$ will be expected to fluctuate with time—and it is just this fluctuation which we wish to study here. In particular,

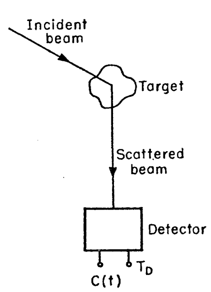

FIG. 1. Illustration of a scattering experiment.

we shall discuss the autocorrelation function

$$G_c(\tau) \equiv \frac{1}{T}\int_0^T C(t+\tau)C(t)dt, \tag{1.2}$$

where the time interval T is so large that [see Eq. (1.1)] $N_c \gg 1$.

We shall assume in this paper that the incident particle beam is "steady" in the sense that its intensity does not systematically drift during the course of the experiment. We shall also assume that the detectors used are 100% efficient, counting every particle which enters them. The fluctuations in $C(t)$ will then be determined by three factors. The first of these is statistical fluctuations in the incident beam intensity, the second is statistical fluctuations in the target, and the third is the transient response characteristic of the detector. The finite response time of the detector will tend to smooth fluctuations in the scattered beam and will thus ordinarily not be welcomed when we are studying fluctuations.

The study of particle beam fluctuations can provide information concerning the structure and degree of coherence of the beam. (For such a study one would of course omit the target and place the detector directly in the incident beam.) After developing the general theory in Sec. II, we shall study beam fluctuations in Sec. III. The results will be applied to an analysis of spectral line broadening of a radiating gas in Sec. IV.

The autocorrelation function (1.2) [more generally, the fluctuations in $C(t)$] can be used to study fluctuations and relaxation processes in the target.[2] Examples of such phenomena, which might be studied with the function (1.2), include density fluctuations in liquids and gases, spin waves in solids, phonon excitations in liquid helium, etc. To apply the theory of Sec. II to such studies, we first (in Sec. VI) cast conventional steady-state scattering theory into such a form that scatterer coordinates are represented by Heisenberg variables at a retarded time. This is then applied in Sec. VII to a

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[1] M. L. Goldberger, H. W. Lewis, and K. M. Watson, Phys. Rev. **132**, 2764 (1963). This paper will henceforth be referred to as I.

[2] That such information is available from fluctuations is evident to anyone who has watched a moving ship or aircraft on a radar A scope.

description of several experiments which might be performed to measure correlations and fluctuations.

In this connection it is perhaps worth noting that temporal fluctuations may be studied even for a target in a pure quantum-mechanical state. For example, let us imagine that we are studying x-ray scattering by hydrogen atoms, each in its ground state. The electron coordinate in a given atom may be written as $z(t)\equiv e^{iht}\times ze^{-iht}$, where h is the atomic Hamiltonian, and its wave function may be written as $g_0(z)$, where $hg_0=w_0g_0$. The average coordinate $(g_0,z(t)g_0)=(g_0,zg_0)$ is of course time-independent. The observation of the autocorrelation function (1.2) provides a measurement of such quantities as $(g_0,\, z(t+\tau)z(t)g_0)$, which *does depend* on τ. Time-dependent motion of this sort is physically meaningful even for pure eigenstates. It is with the observation of such time dependence (for pure states and for statistical mixtures) that we are primarily concerned.

II. CORRELATED COUNTING RATES

In this section we shall express in somewhat simpler form the theory of correlated counting rates as presented in I in the light of the formalism[3] developed in II. In addition we shall rederive some of the results of I without the use of second quantization methods.

We consider an experiment designed to detect particles in a particle beam. These might be particles emerging from an accelerator, or from any kind of radiating source, or they may have been scattered from a target. In any case, we imagine that the experiment lasts for a time interval T, during which n identical beam particles are emitted. If n is sufficiently large, transient effects associated with the beginning or end of the measurement may be ignored.

The time-dependent wave function for the jth beam particle $(j=1,\cdots,n)$ is written as $\Phi_j(\mathbf{x}_j,t)$, where $\mathbf{x}_j$ is the space coordinate of the particle and $(\Phi_j(\mathbf{x}_j,t),\Phi_j(\mathbf{x}_j,t))=1$. Evidently we are using a wave-packet description and not plane waves; this is essential for a proper spatiotemperal discussion. The wave function for the n beam particles is obtained by taking the appropriately symmetrized product of such packets:

$$\Psi(t)=\mathcal{S}\prod_{j=1}^{n}\Phi_j(\mathbf{x}_j,t)\,, \tag{2.1a}$$

where $\mathcal{S}$ is an operator which forms a symmetric wave function for particles satisfying Bose-Einstein (B.E.) statistics and an antisymmetric wave function for particles satisfying Fermi-Dirac (F.D.) statistics.

As is customary, we imagine repeating the experiment many times and represent the effect of this as performing an ensemble average denoted as $\langle\cdots\rangle$. For example, the average density of beam particles at a point $\mathbf{y}$ is

$$\langle\Psi(t),\,\sum_{j=1}^{n}\delta(\mathbf{y}-\mathbf{x}_j)\Psi(t)\rangle=\langle\sum_{j=1}^{n}\Phi_j^*(\mathbf{y},t)\Phi_j(\mathbf{y},t)\rangle; \tag{2.1b}$$

[3] M. L. Goldberger and K. M. Watson, Phys. Rev. **134**, B919 (1964). This paper will henceforth be referred to as II.

where a scalar product of spin and/or other internal variables is implied.

We shall be interested in situations where the ensemble average implies that the number of beam particles n, certain parameters in the Φ_j, and the state of the target (in the case of a scattering problem) are random variables. In particular, we shall be concerned with what we designate as incoherent beams for which the ensemble average implies the following properties[4]:

(1) The random variables describing different beam particles are statistically independent; all beam particles have equivalent statistical properties.

(2) The phases of the various Φ_j are random in a sense made precise in Eq. (3.2). Loosely speaking this phase randomness is associated with the unspecified emission times of particles from a source.

(3) The number of beam particles n is statistically independent of other variables and is described by a Poisson distribution.

(4) The beam is sufficiently uniform that during the course of an experiment, averages such as that of the particle density defined above are independent of time.

(5) The Φ_j may be factored into a product of a space factor Φ_{sj} and a spin factor $u_{\nu j}$. (For scattering experiments a sum of such terms may be required.) Initial spin orientations of the beam particles are random. It follows that the quantity

$$\chi(1)\equiv\langle\Phi_j^*(\mathbf{y}_1,t_1)\Phi_j(\mathbf{y}_1,t_1)\rangle \tag{2.2}$$

is independent of the index j, since all particles are equivalent (assumption 1), and is independent of the time t_1 (assumption 4). We may then write the particle density at $\mathbf{y}_1$ as $\bar{n}\chi(1)$ where

$$\bar{n}=\langle n\rangle\,. \tag{2.3}$$

If the mean speed of beam particles is V, the average particle flux at $\mathbf{y}_1$ is

$$F(\mathbf{y}_1)=\bar{n}V\chi(1)\,. \tag{2.4}$$

We imagine now a particle detector to be located at a point $\mathbf{Y}_1$, a conveniently chosen reference point in the detector. This detector, which we call "one," will be represented by an operator which signifies the presence of particles within the active volume, namely,

$$J_1=\sum_{l=1}^{n}j_l(\mathbf{Y}_1,0)\,, \tag{2.5a}$$

where

$$j_l(\mathbf{Y}_1,0)=\int_1 d^3y\,\gamma_1(\mathbf{y})\delta(\mathbf{y}-\mathbf{x}_l)\,. \tag{2.5b}$$

Here the integral extends over the detector volume and $\gamma_1(\mathbf{y})$ depends on its calibration. For a uniformly sensitive counter, γ_1 is independent of $\mathbf{y}$. Recall also that we

[4] As was done in I, we assume that on taking the ensemble average we may treat the different Φ_j as being effectively orthogonal. The physical implications of this assumption, and generalizations of it, require further study.

assumed in the introduction that our detectors are 100% efficient, giving a count every time a particle enters the active volume.

The instantaneous counting rate of the detector during the experiment is $(\Psi(t),J_1\Psi(t))$; the ensemble average of this,

$$\langle J_1\rangle = \langle(\Psi(t),J_1\Psi(t))\rangle\,, \tag{2.6}$$

is by our assumption independent of time. This may be evaluated in terms of the particle density and detector calibration as described by Eqs. (2.1b), (2.2), and (2.5a,b):

$$\begin{aligned}\langle J_1\rangle &= \int_1 d^3y\gamma_1(\mathbf{y})\langle\sum_{j=1}^{n}\Phi_j^*(\mathbf{y},t)\Phi_j(\mathbf{y},t)\rangle\\ &= \bar{n}\int_1 d^3y_1\gamma_1(y_1)\chi(1)\,,\end{aligned} \tag{2.7}$$

where $\chi(1)$ is the single-particle density function defined by Eq. (2.2). We shall assume that the average beam density (1) is uniform over the detector volume and factor it out of the above integral

$$\langle J_1\rangle = \bar{n}\chi(1)\int_1 d^3y\gamma_1(y)\,. \tag{2.8}$$

The function $\chi(1)$ is evaluated at any convenient point in the detector. For illustrative purposes we occasionally introduce a special counter which is uniform ($\gamma_1=$ constant) and which has flat surfaces of area Σ_1 and thickness w_1 so that

$$\langle J_1\rangle = \bar{n}\chi(1)(w_1\Sigma_1)\gamma_1\,. \tag{2.9}$$

It is also convenient to introduce the concept of a "calibrated counter" defined in such a way that in terms of the average particle flux at $\mathbf{y}_1$ [see Eq. (2.4)]

$$\langle J_1\rangle = \Sigma_1 F(\mathbf{y}_1)\,, \tag{2.10}$$

so for such a counter, the uniform efficiency, γ_1 is given by

$$\gamma_1 = V/w_1\,. \tag{2.11}$$

Now in practice, physical particle counters cannot have the instantaneous response characteristics supposed above. The necessarily finite response time is described by a function $L_1(\tau)$ such that what we call the instantaneous counting rate at a time T_1 is given by[3,5]

$$\int_{-\infty}^{\infty} dt_1 L_1(T_1-t_1)(\Psi(t_1),J_1\Psi(t_1))\,.$$

The causal transient response function $L_1(\tau)$ must satisfy the condition that

$$L_1(\tau)=0 \quad \text{for } \tau<0\,.$$

[5] The linear transient response characteristic has been discussed in more detail in connection with Eq. (4.10) of I and Eq. (2.29) of II.

It is convenient to introduce the frequency characteristic of the detector according to

$$L_1(\tau) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}B_1(\omega)e^{-i\omega\tau}\,, \tag{2.12}$$

and often in practice to imagine that we deal with a low-frequency band pass filter for which

$$B_1(\omega)\approx 1\,,\quad |\omega|<\omega_p\,, \tag{2.13}$$

where ω_p is a frequency such that input signals which have frequencies in the range $-\omega_p<\omega<\omega_p$ are unmodified by the transient response characteristics of the detector.

Taking into account the finite resolving time of the detector, we find for the mean counting rate

$$\begin{aligned}\langle G_1\rangle &= \int_{-\infty}^{\infty} dt\, L_1(T_1-t)\langle J_1\rangle\\ &= B_1(0)\langle J_1\rangle\,,\end{aligned} \tag{2.14a}$$

where the second form follows from the definition of B_1, Eq. (2.12), and the fact that $\langle J_1\rangle$ is independent of time; since $B_1(0)=1$ according to Eq. (2.13) we see that

$$\langle G_1\rangle = \langle J_1\rangle\,. \tag{2.14b}$$

Thus, under the assumptions which led to the time constancy of $\langle J_1\rangle$, we find that the detector response characteristics play no role.

We turn now to the physically more interesting problem posed in Sec. I, namely, the study of fluctuations in the counting rate. In particular, we consider the autocorrelation function defined by Eq. (1.2).[6] Since it involves no added complication we generalize the problem to the study of the space-time correlations of two counters called "one" and "two" located at points $\mathbf{Y}_1$ and $\mathbf{Y}_2$, respectively. The autocorrelation function for a single counter may then be found by setting $\mathbf{Y}_1=\mathbf{Y}_2$ and regarding the two counters as a single counter.

The counting rate operator for detector "two" is given, by analogy with the description of "one," Eq. (2.5), by

$$J_2 = \sum_{l=1}^{n} j_l(\mathbf{Y}_2,0) \tag{2.15a}$$

with

$$j_l(\mathbf{Y}_2,0) = \int_2 d^3y\,\gamma_2(\mathbf{y})\delta(\mathbf{y}-\mathbf{x}_l)\,, \tag{2.15b}$$

where the integral extends over the volume of the second detector and γ_2 is its sensitivity calibration function. It will be convenient to introduce the notion of time-

[6] The theoretical basis for this was developed in II.

dependent, or Heisenberg, counting operators such as

$$J_2(\tau)=\sum_{l=1}^{n} j_l(\mathbf{Y}_2,\tau)\,,$$
$$j_l(\mathbf{Y}_2,\tau)=e^{iK_l\tau}j_l(\mathbf{Y}_2,0)e^{-iK_l\tau}. \qquad (2.16)$$

The quantity K_l is the kinetic energy operator for the lth beam particle.

Now the experiment of interest, illustrated in Fig. 2, is one in which the instantaneous outputs of detectors 1 and 2 are, respectively, voltages $C_1(t)$ and $C_2(t)$, where the voltage C_1 is passed through a distortionless delay line with delay $\tau=\tau_2-\tau_1$ and fed into a correlator C whose output at time t_2 is the product

$$G_{12}=C_2(t_2)C_1(t_2-\tau)\,.$$

The ensemble average, resulting from repeating the experiment many times, is

$$\langle G_{12}\rangle=\langle C_2(t_2)C_1(t_2-\tau)\rangle$$

which under most circumstances of practical importance will be a function of τ only, i.e., independent of t_2.

We consider first the case that the two detectors have a rapid transient response in terms of the scale of the fluctuations. We have shown in II, Eq. (2.21), that for a pure beam state, described by the wave function $\Psi(t_1)$ at the time t_1 of the first measurement that the correlator output is[7]

$$M\equiv\langle J(t_2)J(t_1)\rangle=(\Psi(t_1),\,E_1J_2(t_2-t_1)E_1J_1\Psi(t_1)) \qquad (2.17)$$

for $t_2>t_1$. The meaning of the projection operator E_1 in this expression is

$$\begin{aligned}E_1&=1 \text{ if there } \textit{is} \text{ a particle in detector 1}\\ &=0 \text{ if there is } \textit{no} \text{ particle in detector 1.}\end{aligned} \qquad (2.18)$$

If $t_1>t_2$, the correlation function is[3]

$$\langle J_1(t_1)J_2(t_2)\rangle=(\Psi(t_2),\,E_2J_1(t_1-t_2)E_2J_2\Psi(t_2))\,, \qquad (2.19)$$

where E_2 is the projection operator corresponding to (2.18) for detector 2.

Now we have assumed up to this point that we are dealing with counters which are transparent to beam particles, letting them pass through unimpeded when recording a count. We call such detectors T type. In the other extreme we may imagine using A-type counters which absorb, or otherwise stop the beam particles. The appropriate modifications of our formalism to cover this contingency will be made presently; further the measurement of the autocorrelation function for a single counter will be described.

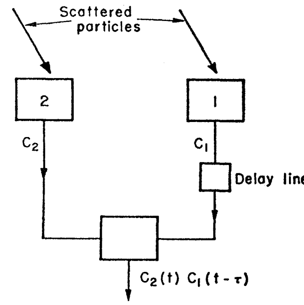

FIG. 2. Correlation of counting rates for two detectors.

When it is necessary to consider the transient response of the detectors, the output of the correlator is taken to be

$$G_{12}=\int_{-\infty}^{\infty}dt_2L_2(T_2-t_2)\int_{-\infty}^{\infty}dt_1L_1(T_1-t_1)$$
$$\times\langle T[J_2(t_2)J_1(t_1)]\rangle, \qquad (2.20)$$

where

$$\begin{aligned}\langle T[J_2(t_2)J_1(t_1)]\rangle&=\langle J_2(t_2)J_1(t_1)\rangle \quad \text{for } t_2>t_1\\ &=\langle J_1(t_1)J_2(t_2)\rangle \quad \text{for } t_1>t_2.\end{aligned} \qquad (2.21)$$

In this case, T_2-T_1 is the delay introduced by the delay line of Fig. 2. The arguments leading to Eq. (2.20) were given in II.

In order to evaluate G_{12} we must make use of the wave function of the system, Eq. (2.1a), and the definitions of the counting operators, J, Eqs. (2.5) and (2.15). We have

$$E_1J_2(\tau)E_1J_1=\sum_{l=1}^{n}E_1j_l(\mathbf{Y}_2,\tau)E_1j_l(\mathbf{Y}_1,0)$$
$$+\sum_{k\neq l=1}^{n}E_1j_k(\mathbf{Y}_2,\tau)E_1j_l(\mathbf{Y}_1,0)\,, \qquad (2.22)$$

where $\tau=t_2-t_1$.

Since $j_l(\mathbf{Y}_1,0)=0$ unless the coordinate $\mathbf{x}_l$ lies in the volume of counter 1, from the definition of E_1, Eq. (2.18), we may set

$$E_1j_l(\mathbf{Y}_1,0)=j_l(\mathbf{Y}_1,0)\,. \qquad (2.23)$$

From this it follows that for $k\neq l$,

$$\begin{aligned}E_1j_k(\mathbf{Y}_2,\tau)j_l(\mathbf{Y}_1,0)&=E_1j_l(\mathbf{Y}_1,0)j_k(\mathbf{Y}_2,\tau)\\ &=j_l(\mathbf{Y}_1,0)j_k(\mathbf{Y}_2,\tau)\\ &=j_k(\mathbf{Y}_2,\tau)j_l(\mathbf{Y}_1,0)\,.\end{aligned} \qquad (2.24)$$

It is clear that for an A-type detector we must set

$$E_1j_l(\mathbf{Y}_2,\tau)E_1j_l(\mathbf{Y}_1,0)=0\,, \qquad (2.25)$$

since if a particle is absorbed in counter 1 it certainly cannot be found in counter 2. We shall assume when we study the autocorrelation function in a single counter

[7] The explicit form derived for Eq. (2.17) in II is

$$M=\sum_s(\psi(t_1),\,E_{1s}J_2(t_2-t_1)E_{1s}J_1\psi(t_1)),$$

where the E_{1s} are projection operators onto the eigenstates of J_1. Here $s(=1,2,\cdots)$ labels a state with one, two, $\cdots$ particles in counter "1." We have replaced this, the correct expression, by the simpler quantity, (2.17), where in terms of the above E_{1s}, $E_1=\sum_sE_{1s}$. For all of the applications in the present paper, the two forms are to a good approximation equivalent. We shall discuss this point in detail in a forthcoming paper on coherent beam fluctuations.

that a single particle cannot be counted twice, so that the condition (2.25) obtains in this case also.

We are finally in a position to compute the expectation value in the state $\Psi(t_1)$ described by Eq. (2.1a) of the relevant product of the J's as described by Eq. (2.17). We consider first $t_2 > t_1$ and recall the definition of $j_k(\mathbf{Y}_2,\tau)$, Eq. (2.16), and obtain after an elementary calculation

$$M = M_n(t_2,t_1) + M_d(t_2,t_1)$$

where

$$\begin{aligned} M_n(t_2,t_1) &\equiv \Big(\Psi(t_1), \sum_{k\neq l=1}^{n} j_k(\mathbf{Y}_2,\tau) J_l(\mathbf{Y}_1,0)\Psi(t_1)\Big) \\ &= \int_1 d^3y_1\gamma_1(\mathbf{y}_1)\int_2 d^3y_2\gamma_2(\mathbf{y}_2)\sum_{k\neq l=1}^{n}\big[\Phi_k^*(\mathbf{y}_2,t_2) \\ &\quad\times \Phi_k(\mathbf{y}_2,t_2)\Phi_l^*(\mathbf{y}_1,t_1)\Phi_l(\mathbf{y}_1,t_1) \pm \Phi_k^*(\mathbf{y}_2,t_2) \\ &\quad\times \Phi_l(\mathbf{y}_2,t_2)\Phi_l^*(\mathbf{y}_1,t_1)\Phi_k(\mathbf{y}_1,t_1)\big], \end{aligned} \tag{2.26}$$

and

$$M_d(t_2,t_1) \equiv \Big(\Psi(t_1), \sum_{l=1}^{n} E_1 j_l(\mathbf{Y}_2,\tau) j_l(\mathbf{Y}_1,0)\Psi(t_1)\Big). \tag{2.27}$$

The plus or minus sign in Eq. (2.26) refers to B.E. or F.D. statistics, respectively. Evidently the M_n comes from the second term of (2.22), simplified according to (2.24), whereas M_d comes from the first term using Eq. (2.23).

For T-type detectors we have both terms contributing to M, whereas for A-type or for the single detector autocorrelation experiment we have only M_n, according to the condition (2.25) which forbids in either of these cases the counting twice of the same particle. To summarize:

$$\begin{aligned} M &= M_n(t_2,t_1) + M_d(t_2,t_1), && T\text{-type}, \\ M &= M_n(t_2,t_1), && A\text{-type}, \\ M &= M_n(t_2,t_1), && \text{autocorrelation.} \end{aligned} \tag{2.28}$$

It is necessary at this point to establish some conventions for carrying out spin sums and averages of our expressions for M, Eqs. (2.28). Since $\Psi(t_1)$ represents a pure state, the various beam particles have a definite spin orientation. When the ensemble average is performed we must average over these orientations. The single-particle wave function Φ (Φ^*) represents column (row) matrices, and matrix products between adjacent Φ^* and Φ are implied in Eqs. (2.26) and (2.27). In a scattering experiment, the scattering amplitude contained in Φ will be a column matrix for each initial spin orientation[8] and an average over these will eventually be carried out.

When we may replace M by M_n,[9] the time ordering of the product of the counting operators defined by Eqs. (2.17) and (2.18) is irrelevant, since the operators commute. The correlation function, Eq. (2.20), becomes simply

$$G_{12} = \int_{-\infty}^{\infty} dt_2 L_2(T_2 - t_2) \times \int_{-\infty}^{\infty} dt_1 L_1(T_1 - t_1) M_n(t_2,t_1). \tag{2.29}$$

It is convenient to introduce an abbreviated notation:

$$\int (1)\cdots \equiv \int_{-\infty}^{\infty} dt_1 L_1(T_1 - t_1)\int_1 d^3y_1\gamma_1(\mathbf{y}_1)\cdots, \tag{2.30}$$

$$\Phi_k(1) \equiv \Phi_k(\mathbf{y}_1,t_1),$$

and similarly for things labeled with "2." We may then rewrite G_{12}, using the definition of M_n, Eq. (2.26), as

$$G_{12} = \sum_{k\neq l=1}^{n}\big[A_{kl} \pm B_{kl}\big], \tag{2.31}$$

where

$$A_{kl} = \int (1)\int (2)\Phi_k^*(1)\Phi_k(1)\Phi_l^*(2)\Phi_l(2), \tag{2.32}$$

$$B_{kl} = \int (1)\int (2)\Phi_k^*(2)\Phi_l(2)\Phi_l^*(1)\Phi_k(1). \tag{2.33}$$

The ensemble average now yields the correlation function

$$\begin{aligned} \langle G_{12}\rangle &= \Big\langle \sum_{k\neq l=1}^{n}\big[A_{kl}\pm B_{kl}\big]\Big\rangle \\ &= \langle n(n-1)\rangle\big[\langle A_{kl}\rangle \pm \langle B_{kl}\rangle\big] \\ &= \bar{n}^2\big[\langle A_{kl}\rangle \pm \langle B_{kl}\rangle\big], \end{aligned} \tag{2.34}$$

where $k\neq l$ and we have used our assumption 3 of a Poisson distribution of particle numbers to replace $\langle n(n-1)\rangle$ by $\bar{n}^2$. Now, by assumption 1 on the statistical independence of the beam particles, we may write

$$\langle \Phi_k^*(1)\Phi_k(1)\Phi_l^*(2)\Phi_l(2)\rangle = \langle \Phi_k^*(1)\Phi_k(1)\rangle\langle \Phi_l^*(2)\Phi_l(2)\rangle$$
$$\langle \Phi_k^*(2)\Phi_l(2)\Phi_l^*(1)\Phi_k(1)\rangle g = \langle \Phi_k^*(2)\Phi_k(1)\rangle\langle \Phi_l^*(1)\Phi_l(2)\rangle,$$

where g is a factor taking account of the average over spin orientations [see Eq. (2.38) below], and define

$$\chi(12) = \langle \Phi_l^*(1)\Phi_l(2)\rangle = \chi^*(21); \tag{2.35}$$

we recall the definition $\chi(1) = \langle \Phi_k^*(1)\Phi_k(1)\rangle$, Eq. (2.2), and write

$$\langle G_{12}\rangle = \bar{n}^2\int (1)\int (2)\big[\chi(1)\chi(2) \pm g|\chi(12)|^2\big]. \tag{2.36}$$

Finally, we note that the first term involves the product of the mean counting rates in the individual

[8] It will also in general be a function of the target particle spin operators, implying further matrix products in these variables.

[9] Even for T-type detectors, the term M_d will be negligibly small except when they are arranged in line with the beam, as in a counter telescope. This is shown in Sec. V.

counters [see Eqs. (2.7) and (2.14a)] so that we have

$$\langle G_{12}\rangle=\langle G_1\rangle\langle G_2\rangle\pm g\bar{n}^2\int(1)\int(2)|\chi(12)|^2. \quad (2.37)$$

For a completely polarized beam, $g=1$. For a randomly polarized beam,

$$g=\langle\delta\nu_k,\nu_l\rangle\,, \quad (2.38)$$

where the average is taken over the initial spin orientations ν_k and ν_l of particles k and l. For an unpolarized beam of electrons or photons we have

$$g=\tfrac{1}{2}. \quad (2.39)$$

Equation (2.37) represents the major result of this section. It is equivalent to the corresponding result obtained in I, but stated there somewhat less generally in that we had set $T_2=T_1$.

III. APPLICATION TO AN INCOHERENT BEAM

In this section we shall apply our fundamental expression for $\langle G_{12}\rangle$, Eq. (2.37), to the discussion of an incoherent beam. This might be a beam emitted by a radiating or radioactive source, or by an accelerator. In any of these we write a typical beam particle wave function as $(j=1, 2,\cdots, n)$

$$\Phi_j(\mathbf{x},t)=\frac{1}{x}\int dq a_j(q)e^{i(qx-\epsilon_q t)}\,, \quad (3.1)$$

if we omit the spin wave function. Here $x=|\mathbf{x}|$, and ϵ_q is the energy of a particle with momentum q and $a_j(q)$ is a wave packet amplitude. The phase incoherence assumption for the beam, called assumption 2 in Sec. II is to be interpreted to mean that for the ensemble average

$$\langle a_j(p)a_k{}^*(q)\rangle=\delta_{jk}\delta(p-q)\rho(q)\langle|a_j(q)|^2\rangle. \quad (3.2)$$

Here $\rho(q)$ is a weight function which we shall now evaluate.[10]

At a distance y_1 from the source we write the particle flux, Eq. (2.4), as

$$F(\mathbf{y}_1)=\bar{n}V\chi(1)=R_B/4\pi y_1{}^2\,, \quad (3.3)$$

where R_B is the equivalent isotropic source strength. If we substitute the wave-packet representation for Φ_j into the definition of $\chi(1)$, namely $\chi(1)=\langle\Phi_j(1)\Phi_j(1)\rangle$, Eq. (2.2), we find

$$\chi(1)=\frac{1}{y_1{}^2}\int dq\rho(q)\langle|a_j(q)|^2\rangle. \quad (3.4)$$

It is convenient to express q in terms of the particle energy by writing $\omega=\epsilon_q$ and introducing the energy spectrum of the beam, $g(\omega)$, as follows: Define

$$dq\rho(q)\langle|a_j(q)|^2\rangle\equiv(N_B/\bar{n})g(\omega)d\omega\,, \quad (3.5)$$

[10] A somewhat different description of the ensemble average (3.2) was given in I.

where

$$\int d\omega g(\omega)=1\,, \quad (3.6)$$

and N_B is a constant. We find immediately from Eq. (3.3) and these definitions

$$N_B=R_B/4\pi V\,, \quad (3.7)$$

which has an obvious interpretation.

It is now easy to express the important correlation function $\chi(12)$ defined by Eq. (2.35) in terms of the spectral function $g(\omega)$. For the special case of a point source we obtain

$$\chi(12)=\frac{N_B}{\bar{n}y_1y_2}\int d\omega\, g(\omega)e^{i[q(y_2-y_1)-\omega(t_2-t_1)]}\,, \quad (3.8)$$

where q is implicitly defined in terms of ω by $\omega=\epsilon_q$. We may complete the evaluation of $\langle G_{12}\rangle$, Eq. (2.37), using this result for $\chi(12)$ and the Fourier representation for the transient[2] response functions $L(T-t)$ given by Eq. (2.12). We obtain

$$\bar{n}^2\int(1)\int(2)|\chi(12)|^2=N_B{}^2\int_1\frac{d^3y_1\gamma_1(\mathbf{y}_1)}{y_1{}^2}$$
$$\times\int\frac{d^3y_2\gamma_2(\mathbf{y}_2)}{y_2{}^2}\int d\omega\int d\omega' g(\omega)g(\omega')B_1(\omega'-\omega)$$
$$B_2(\omega-\omega')e^{i[(q-q')(y_2-y_1)-(\omega-\omega')(T_2-T_1)]} \quad (3.9)$$

where $\omega'=\epsilon_{q'}$.

In the limit that the bandwidth function B is very narrow compared to the beam-spectral function $g(\omega)$, we may approximate (3.9) as follows (we take $B_1=B_2$ for simplicity);

$$\bar{n}^2\int(1)\int(2)|\chi(12)|^2$$
$$\cong\frac{N_B{}^2}{\Delta\omega_B}\int_1\frac{d^3y_1\gamma_1(\mathbf{y}_1)}{y_1{}^2}\int\frac{d^3y_2\gamma_2(\mathbf{y}_2)}{y_2{}^2}\int d\omega|B(\omega)|^2$$
$$\times\exp\left\{i\left[\omega(T_1-T_2)-\frac{1}{V}(y_1-y_2)\right]\right\}\,, \quad (3.10)$$

where $V=d\omega/dq$, evaluated at the mean beam energy, and the beam spectral width $\Delta\omega_B$ is defined by

$$(\Delta\omega_B)^{-1}\equiv\int d\omega[g(\omega)]^2. \quad (3.11)$$

The other limit of interest is one in which the bandwidth function is very broad compared to the variation of the beam-spectral function. In this case we may replace $B(\omega-\omega')$ by $B(0)$ since by hypothesis during the maximum excursion of $\omega-\omega'$, B scarcely varies. We find

$$\bar{n}^2\int(1)\int(2)|\chi(12)|^2\cong N_B{}^2B_1(0)B_2(0)\times\int_1\frac{d^3y_1\gamma_1(\mathbf{y}_1)}{y_1{}^2}\int\frac{d^3y_2\gamma_2(\mathbf{y}_2)}{y_2{}^2}\left|\int d\omega g(\omega)\times\exp\{i[q(y_2-y_1)-\omega(T_2-T_1)]\}\right|^2. \quad (3.12)$$

We have thus completed the evaluation of the correlated counting rate $\langle G_{12}\rangle$ for two cases of practical importance. Needless to say, if neither the narrow- nor the broad-band conditions obtain, the complete result, Eq. (3.9), must be used. Finally, as noted in I, in the case of narrow electronic bandwidth (i.e., narrow with respect to the beam-spectral function), it may be advantageous to put a dc blocking filter in the detector output. This means, according to Eq. (2.14a), that $\langle G_1\rangle=0$ [and, of course, $\langle G_2\rangle=0$] since $B_1(0)=B_2(0)=0$ if no dc is passed. Thus $\langle G_{12}\rangle$ will be given by (3.9) or (3.10).

The assumption of a point source made in obtaining Eq. (3.8) must in general be relaxed for macroscopic incoherent sources. This is easily done by merely averaging the expression (3.8) over source points. To do this we let $\mathbf{y}_1(\mathbf{y}_2)$ be a vector from a fixed point $\mathcal{O}$ in the source to some point in detector "1" ("2") and $\mathbf{s}$ be a vector from $\mathcal{O}$ to a point in the source and define

$$\begin{aligned}\mathbf{D}_1(\mathbf{s})&\equiv\mathbf{y}_1-\mathbf{s},\\ \mathbf{D}_2(\mathbf{s})&\equiv\mathbf{y}_2-\mathbf{s}.\end{aligned} \quad (3.13)$$

Then, for a source of uniform intensity

$$\chi(12)=\frac{N_B}{\bar{n}}\int_s\frac{d^3s}{\mathcal{V}_cD_1(\mathbf{s})D_2(\mathbf{s})}d\omega g(\omega)\times\exp\{i[q(D_2(\mathbf{s})-D_1(\mathbf{s}))-\omega(t_2-t_1)]\}. \quad (3.14)$$

Here the integral on $\mathbf{s}$ extends over the volume (or surface) $\mathcal{V}_s$ of the source. (The appropriate modification for a source of nonuniform intensity is obvious.)

When the linear dimensions of the source are small compared with y_1 and y_2 we may rewrite Eq. (3.14) in the simpler form

$$\chi(12)=\frac{N_B}{\bar{n}y_1y_2}\int d\omega\, g(\omega)\exp\{i[q(y_2-y_1)-\omega(t_2-t_1)]\}Q(q;\hat{y}_1-\hat{y}_2), \quad (3.15)$$

where $\hat{y}\equiv\mathbf{y}/y$ and

$$Q(q;\hat{y}_1-\hat{y}_2)\equiv\int_s\frac{d^3s}{\mathcal{V}_s}\exp[iq(\hat{y}_1-\hat{y}_2)\cdot\mathbf{s}]. \quad (3.16)$$

When the beam spectral width is narrow enough that q may be replaced by a mean momentum p in Q we can rewrite Eq. (3.15) as

$$\chi(12)=\chi_P(12)Q(p;\hat{y}_1-\hat{y}_2). \quad (3.17)$$

Here, $\chi_P(12)$ is the value of $\chi(12)$, as given by Eq. (3.8), for a point source of the same intensity as that for which $\chi(12)$ is calculated.

The illustrations given in the next two sections, assuming point sources, are rather trivially modified for a finite source when Eq. (3.17) is used. This will be done in a subsequent paper in which we shall study counting accuracy and observation times in detail.

IV. APPLICATION TO A RADIATING SOURCE

We shall now show that the formalism developed in in Sec. III can be used for the analysis of spectral lines.[11] For the sake of illustration we consider a radiating source which has a Lorentz line shape. (Similar considerations apply to the observation of the spectrum from a radioactive source.) The wave function of a typical beam particle has the form[12] (for $j=1, 2,\cdots, n$)

$$\begin{aligned}\Phi_j(\mathbf{x},t)&=\frac{C}{x}e^{i[px-\epsilon_p(t-t_j)]}e^{-\frac{1}{2}\Gamma[t-t_j-(x/V)]}\quad\text{for}\quad t-t_j-(x/V)>0\\ &=0\qquad\text{for}\quad t-t_j-(x/V)<0.\end{aligned} \quad (4.1)$$

The linewidth is Γ, t_j is the "emission time," C is a normalization constant, and V is the particle velocity which is, of course, c, the velocity of light in the case of photons.

The wave packet amplitude $a_j(q)$, defined by the Fourier transform of (4.1), according to Eq. (3.1) is readily found to be

$$a_j(q)=-\frac{CV}{2\pi i}\frac{e^{i\epsilon_qt_j}}{\epsilon_q-\epsilon_p+(i\Gamma/2)}. \quad (4.2)$$

The beam spectral function $g(\omega)$ is therefore

$$g(\omega)=\frac{\Gamma}{2\pi}\left[(\omega-\epsilon_p)^2+\frac{\Gamma^2}{4}\right]^{-1}. \quad (4.3)$$

To find the correlation function we must evaluate Eq. (3.9). The analysis is particularly simple for the case of electromagnetic radiation in the visible or lower frequency part of the spectrum. The counters may be made small enough that they be regarded as point counters at $\mathbf{Y}_1$ and $\mathbf{Y}_2$, and the electronic response times made fast enough to give credibility to the broad-band approximation, Eq. (3.12). With the beam spectral function given by (4.3) we find immediately [here

[11] An application of the Hanbury Brown-Twiss technique for this purpose was suggested by A. T. Forrester, J. Opt. Soc. Am. **51**, 253 (1961).

[12] See Eq. (4.1) in II or, for example, Eq. (8-119) of M. L. Goldberger and K. M. Watson, in *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

g is the spin average factor (2.38)]

$$\langle G_{12}\rangle = \langle G_1\rangle\langle G_2\rangle \times\{1+g\exp[-\Gamma|T_2-T_1-(1/c)(Y_2-Y_1)|]\}. \quad (4.4)$$

The autocorrelation function for a single counter is obtained by setting $\langle G_2\rangle = \langle G_1\rangle$ and $Y_2=Y_1$:

$$\langle G_c\rangle = \langle G_1\rangle^2\{1+g\exp(-\Gamma|T_2-T_1|\}. \quad (4.5)$$

Thus, the linewidth, Γ can be determined by a measurement of $\langle G_{12}\rangle$ or $\langle G_c\rangle$, the latter being sufficient and slightly simpler experimentally.

V. EVALUATION OF M_d

We have up to this point in our evaluation of the correlated counting rate $\langle G_{12}\rangle$ neglected the contribution from events which correspond to the same particle passing through both counters. This contribution to $\langle G_{12}\rangle$ which is, of course, relevant only for the case of two T-type detectors, was called M_d in Sec. II, Eq. (2.27). We shall discuss briefly the evaluation of M_d.

For simplicity let us assume that the two counters are identically constructed as thin flat disks, each of surface area Σ and thickness w; further we take them to be spatially uniform so that the quantities $\gamma_1(\mathbf{y}_1)=\gamma_2(\mathbf{y})=\gamma$ are constant [see Eq. (2.5b)]. Each counter is to be oriented so that the beam impinges normally on its flat surface. We shall consider only the case of a beam of low enough density that when the projection operator E_1 acts on $j_l(\mathbf{Y}_2,\tau)$ we may write

$$E_1\approx e_l, \quad (5.1)$$

where

$$e_l=\int_1 d^3y\,\delta(\mathbf{y}-\mathbf{x}_l). \quad (5.2)$$

This is an approximation in that E_1 is defined to be unity if *any* particle is in counter one; we have already assumed that the lth particle is in the counter or else the factor $j_l(\mathbf{Y}_1,0)$ which enters M_d, Eq. (2.27) vanishes. There could, however, be other particles, beside l, in the counter. We shall discuss briefly at the end of this section the opposite limit of a very high density beam.

From the definition of M_d, Eq. (2.27), and the wave function for the system, Eq. (2.1), we find

$$M_d=\sum_{l=1}^{n}(\Phi_l(\mathbf{x}_l,t_1),\, e_l j_l(\mathbf{Y}_2,\tau)j_l(\mathbf{Y}_1,0)\Phi_l(\mathbf{x}_l,t_1))$$
$$=\gamma^2\sum_{l=1}^{n}\int d^3x\,\Phi_l^*(\mathbf{x},t_1)$$
$$\times\int_1 d^3y_1'\delta(\mathbf{y}_1'-\mathbf{x})\int_2 d^3y_2 e^{iK\tau}\delta(\mathbf{y}_2-\mathbf{x})e^{-iK\tau}$$
$$\times\int_1 d^3y_1\delta(\mathbf{y}_1-\mathbf{x})\Phi_l(\mathbf{x},t_1). \quad (5.3)$$

Here K is the kinetic energy operator for the lth beam particle which we take to have the form

$$K=P^2/2M,$$

where $\mathbf{P}=-i\nabla_x$ is the momentum operator, and M is the particle mass. Now repeatedly using the well-known relation[13]

$$e^{i\tau\nabla^2/2M}f(\mathbf{x})=\int d^3x' e^{i\tau\nabla^2/2M}\delta(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')$$
$$=(M/2\pi i\tau)^{3/2}\int d^3x'$$
$$\times\exp[iM(\mathbf{x}-\mathbf{x}')^2/2\tau]f(\mathbf{x}'), \quad (5.4)$$

we find from Eq. (5.3)

$$M_d=\sum_{l=1}^{n}\left(\frac{M}{2\pi\tau}\right)^3\gamma^2\int_1 d^3y_1'\int_1 d^3y_1\int_2 d^3y_2$$
$$\times\Phi_l^*(\mathbf{y}_1',t_1)\Phi_l(\mathbf{y}_1,t_1)\exp\left[i\frac{M}{2\tau}(\mathbf{y}_1-\mathbf{y}_1')^2\right]$$
$$\times\exp\left[-i\frac{M}{\tau}(\mathbf{y}_1'-\mathbf{y}_1)\cdot(\mathbf{y}_1'-\mathbf{y}_2)\right]. \quad (5.5)$$

If we now substitute the Fourier representation of the wave functions Φ_l together with Eq. (3.8), used earlier, we obtain for the ensemble average of M_d the result

$$\langle M_d\rangle=\left(\frac{M}{2\pi\tau}\right)^3\gamma^2N_B$$
$$\times\int\frac{d^3y_1'}{y_1'}\int\frac{d^3y_1}{y_1}\int d^3y_2\exp\left[i\frac{M}{2\tau}(\mathbf{y}_1-\mathbf{y}_1')^2\right]$$
$$\times\exp\left[-i\frac{M}{\tau}(\mathbf{y}_1'-\mathbf{y}_1)\cdot(\mathbf{y}_1'-\mathbf{y}_2)\right]$$
$$\times\int d\omega\,g(\omega)\exp[iq(\omega)(y_1-y_1')], \quad (5.6)$$

where $q(\omega)$ is the momentum of a beam particle of energy, ω.

As a simple illustration of Eq. (5.6), let us suppose that the counters have dimensions small compared with their separation from the source. We introduce new variables

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{y}_1'+\mathbf{y}_1),$$
$$\mathbf{r}=\mathbf{y}_1'-\mathbf{y}_1,$$

and assume $|\mathbf{r}|\ll|\mathbf{R}|\approx Y_1$, the distance of counter 1

[13] We have here used the relation $e^ae^b=\exp[\frac{1}{2}(a,b)]\exp(a+b)$, valid when $[a,[a,b]]=[b,[a,b]]=0$.

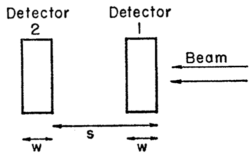

FIG. 3. A simple counter telescope illustrating the measurement implied by Eq. (5.9).

from the source. We obtain

$$\langle M_d\rangle = \frac{N_B\gamma^2}{Y_1^2}\int d\omega\, g(\omega)$$

$$\times \int d^3R \int d^3y_2 \delta\left[\mathbf{R}-\mathbf{y}_2+\frac{\tau}{M}q\hat{R}\right], \quad (5.7)$$

where $\hat{R}=\mathbf{R}/R$.

To simplify this expression further, let us assume that the beam spectrum is very narrow and centered at a frequency corresponding to momentum p, and let $q=p$ be set inside the δ function; the integral over ω then, from the normalization of $g(\omega)$, is unity and we have

$$\langle M_d\rangle \approx \frac{N_B\gamma^2}{Y_1^2}\int d^3R \int d^3y_2 \delta\left[\mathbf{R}-\mathbf{y}_2+\frac{\tau}{M}p\hat{R}\right]. \quad (5.8)$$

The physical interpretation of Eq. (5.8) is clear: If the two counters are aligned with the beam direction and if τ is the flight time between the counters, a coincidence will occur. If the two counters are aligned exactly, one directly behind the other a distance s apart (illustrated in Fig. 3), and if we use the expression $\gamma=V/w$, Eq. (2.11), we find

$$\langle M_d\rangle = \langle G_1\rangle \frac{V}{w^2}\times \begin{cases} 0, & \text{if } \tau p/M < s-w \\ w-s+\tau p/M, & \text{if } s-w<\tau p/M<s \\ w+s-\tau p/M, & \text{if } s<\tau p/M<s+w \\ 0, & \text{if } \tau p/M > s+w. \end{cases} \quad (5.9)$$

Finally we consider the limiting case of a very dense beam. It should be evident that the projection operator E_1 plays no role in the expression for M_d and we may assume that there is always at least one particle in counter one; hence $E=1$. We have then,

$$M_d=\sum_{l=1}^{n}(\Phi_l(\mathbf{x}_l,t_1),\, j_l(\mathbf{Y}_2,\tau)j_l(\mathbf{Y}_1,0)\Phi_l(\mathbf{x}_l,t_1))$$

$$=\gamma^2\sum_{l=1}^{n}\int d^3x\, \Phi_l^*(\mathbf{x},t_1)\int_2 d^3y_2 e^{iK\tau}\delta(\mathbf{y}_2-\mathbf{x})e^{-iK\tau}$$

$$\times\int_1 d^3y_1\delta(\mathbf{y}_1-\mathbf{x})\Phi_l(\mathbf{x},t_1). \quad (5.10)$$

Using Eq. (5.4), as before, we find

$$M_d=\left(\frac{M}{2\pi\tau}\right)^3\gamma^2\sum_{l=1}^{n}\int d^3x\int_1 d^3y_1$$

$$\times\int_2 d^3y_2\Phi_l^*(\mathbf{x},t_1)\Phi_l(\mathbf{y}_1,t)\exp\left[-\frac{iM(\mathbf{x}-\mathbf{y}_2)^2}{2\tau}\right]$$

$$\times\exp\left[\frac{iM(\mathbf{y}_2-\mathbf{y}_1)^2}{2\tau}\right]. \quad (5.11)$$

We now form the ensemble average of M_d and use the expression (3.8) for the average of the Φ's to obtain

$$\langle M_d\rangle=\left(\frac{M}{2\pi\tau}\right)^3\gamma^2N_B\int\frac{d^3x}{x}\int_1\frac{d^3y_1}{y_1}\int d^3y_2$$

$$\times\exp\left[\frac{-iM(\mathbf{x}-\mathbf{y}_2)^2}{2\tau}\right]\exp\left[\frac{iM(y_2-y_1)^2}{2\tau}\right]$$

$$\times\int d\omega\, g(\omega)\exp[iq(\omega)(y_1-x)].$$

VI. TIME-DEPENDENT SCATTERING THEORY

We turn now to the major problem studied in this paper, namely, the analysis of fluctuations in a beam of scattered particles and the question of what information about the scatterer can be deduced from such a study. To do this, we must first cast conventional scattering theory into a form which is suitable for exhibiting the time dependence of the scattering process.

We begin with the description of scattering of a single beam particle by a single, bound, scattering particle. This will be done in the impulse approximation in which the scattering matrix is replaced by that for a free particle with a momentum distribution characteristic of the initially bound state.

The bound system is described by a Hamiltonian h and wave function $g_0(\mathbf{z})$,[14] where $\mathbf{z}$ is the particle coordinate. Thus

$$hg_0(\mathbf{z})=w_0g_0(\mathbf{z}), \quad (6.1)$$

where $-w_0$ is the binding energy. The momentum of the beam particle prior to the scattering is $\mathbf{p}$ so that the initial configuration of the system beam particle and bound scatterer may be written as

$$\chi_a=(2\pi)^{-3/2}e^{i\mathbf{p}_1\cdot\mathbf{x}}ug_0(\mathbf{z}), \quad (6.2)$$

where u is the spin wave function of the projectile. The scattered particle wave function is then, if K is the beam particle kinetic energy operator and $\mathcal{T}$ the scat-

[14] To avoid a distracting notation, we continue to avoid exhibiting spin variables explicitly.

tering matrix,[15]

$$\psi_{sc}=(w_0+\epsilon_p+i\eta-h-K)^{-1}\mathcal{T}\chi_a$$

$$=-i\int_0^\infty d\tau\exp[i(w_0+\epsilon_p+i\eta-K-h)\tau]\mathcal{T}\chi_a. \quad (6.3)$$

The time-dependent scattered wave function for a time long after the scattering is completed may be obtained from Eq. (6.3) by multiplication by $\exp[-i(h+K)t]$ and replacing the lower limit of integration by $-\infty$. Although we shall ultimately require ψ_{sc} for such long times, we shall continue to use the representation (6.3) which gives the scattered wave function exactly for all points in configuration space, not just in the asymptotic region.

Since $\mathcal{T}$ is the two-particle scattering matrix, we have for a free particle of momentum $\mathbf{Q}_0$

$$\mathcal{T}\exp(i\mathbf{p}\cdot\mathbf{x})\exp(i\mathbf{Q}_0\cdot\mathbf{z})$$

$$=\int d^3k\int d^3Q\exp[i(\mathbf{k}\cdot\mathbf{x}+\mathbf{Q}\cdot\mathbf{z})]\times\delta(\mathbf{k}+\mathbf{Q}-\mathbf{p}-\mathbf{Q}_0)(\mathbf{k},\mathbf{Q}|T|\mathbf{p},\mathbf{Q}_0)$$

$$=\int d^3k\exp[i(\mathbf{k}\cdot\mathbf{x}-\boldsymbol{\varrho}\cdot\mathbf{z})]\times(\mathbf{k},\mathbf{Q}_0-\boldsymbol{\varrho}|T|\mathbf{p},\mathbf{Q}_0)\exp(i\mathbf{Q}_0\cdot\mathbf{z}), \quad (6.4)$$

where $\boldsymbol{\varrho}=\mathbf{k}-\mathbf{p}$, the momentum transfer to the scatterer and T is the submatrix of $\mathcal{T}$ on the momentum shell. If we now introduce a Fourier representation of g_0,

$$g_0(\mathbf{z})=(2\pi)^{-3/2}\int d^3Q_0\exp(i\mathbf{Q}_0\cdot\mathbf{z})a(\mathbf{Q}_0), \quad (6.5)$$

we find for $\mathcal{T}\chi_a$ the result

$$\mathcal{T}\chi_a=(2\pi)^{-3/2}\int d^3k\exp[i(\mathbf{k}\cdot\mathbf{x}-\boldsymbol{\varrho}\cdot\mathbf{z})]\times(\mathbf{k},\mathbf{Q}_0-\boldsymbol{\varrho}|T|\mathbf{p},\mathbf{Q}_0)ug_0(\mathbf{z}), \quad (6.6)$$

where we must interpret $\mathbf{Q}_0$ as $\mathbf{Q}_0=-i\nabla_z$. Finally, we insert this into our formula for the scattered wave, Eq. (6.3), and obtain

$$\psi_{sc}=-(2\pi)^{-3/2}i\int_0^\infty d\tau\int d^3k\times\exp[i(\epsilon_p+i\eta-\epsilon_k)\tau]\exp\{i[\mathbf{k}\cdot\mathbf{x}-\boldsymbol{\varrho}\cdot\mathbf{z}(-\tau)]\}(\mathbf{k},\mathbf{Q}_0(-\tau)-\boldsymbol{\varrho}|T|\mathbf{p},\mathbf{Q}_0(-\tau))ug_0(\mathbf{z}), \quad (6.7)$$

where $e^{-ih\tau}$ has been commuted to the right to act on g_0, thereby canceling the factor $e^{iw_0\tau}$ in Eq. (6.3). We have also indicated explicitly the concomitant time dependence in $\mathbf{z}$ and $\mathbf{Q}_0$ according to the definitions

$$\mathbf{z}(-\tau)=e^{-ih\tau}\mathbf{z}e^{ih\tau}, \quad (6.8)$$

etc.

We require ψ_{sc} only for large x, far from the scatter, $x\gg1/p$. In this limit we may as usual integrate over the directions of $\mathbf{k}$ to obtain[16]

$$\psi_{sc}=-(2\pi)^{-1/2}x^{-1}\int_0^\infty d\tau\int_0^\infty kdk\,e^{iP(k,\tau)}Tug_0(\mathbf{z}), \quad (6.9)$$

where

$$P(k,\tau)=(\epsilon_p+i\eta-\epsilon_k)\tau+kx-\boldsymbol{\varrho}\cdot\mathbf{z}(-\tau), \quad (6.10)$$

and now (we use the notation $\hat{x}\equiv\mathbf{x}/x$)

$$\boldsymbol{\varrho}=k\hat{x}-p. \quad (6.11)$$

We may simplify the expression for ψ_{sc}, Eq. (6.9), in several limiting cases. We consider the weak binding limit[15] in which the $\mathbf{Q}_0$ dependence of T may be neglected and also suppose the target particle to be so heavy that its recoil from the collision can be neglected. In this case we may write

$$\psi_{sc}=\frac{e^{ipx}}{(2\pi)^{3/2}x}\mathbf{f}g_0, \quad (6.12)$$

where the scattering amplitude $\mathbf{f}$

$$\mathbf{f}=-2\pi\int_0^\infty d\tau\int_0^\infty kdk\exp\{i[P(k,\tau)-px]\}Tu \quad (6.13)$$

is a column matrix in the beam particle spin variables and a square matrix in those of the target particle. To evaluate the integrals, we set $k=p+q$, where q is considered to be small and expand the exponent in (6.13) about $q=0$:

$$P(k,\tau)-px=i\eta\tau-\boldsymbol{\varrho}_0\cdot\mathbf{z}(-\tau)+q[-V\tau+x-\hat{x}\cdot\mathbf{z}(-\tau)]+\cdots,$$

where

$$V=d\epsilon_p/dp,$$

$$\boldsymbol{\varrho}_0=p\hat{x}-p.$$

The integration over q may be carried out approximately by writing

$$\int_0^\infty kdk\cdots=\int_{-p}^\infty(p+q)dq\cdots\approx p\int_{-\infty}^\infty dq\cdots$$

which is legitimate since we are in the asymptotic region where $px\gg1$. We find

$$\mathbf{f}=-(2\pi)^2pTu\int d\tau\,\delta[x-V\tau-\hat{x}\cdot\mathbf{z}(-\tau)]\times\exp[-i\boldsymbol{\varrho}_0\cdot\mathbf{z}(-\tau)]$$

$$=-(2\pi)^2p\frac{T}{V}u\frac{\exp[-i\boldsymbol{\varrho}_0\cdot\mathbf{z}(-\tau_r)]}{1-\hat{x}\cdot\mathbf{z}(-\tau_r)/V}, \quad (6.14)$$

where $(-\tau_r)$ is the retarded time defined by

$$\tau_r=\frac{1}{V}[x-\hat{x}\cdot\mathbf{z}(-\tau_r)]. \quad (6.15)$$

[15] We are following the notation of Chap. II, *Collision Theory*, Ref. 12.

[16] That is, we take $\int d\Omega_k e^{i\mathbf{k}\cdot\mathbf{x}}F(\hat{k})=(2\pi/ikx)F(\hat{x})e^{ikx}$ in the integrand of Eq. (6.7).

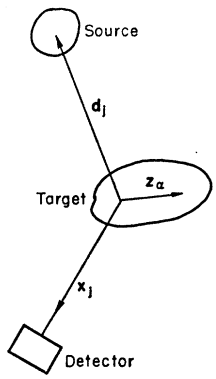

FIG. 4. Scattering by a composite target.

Since we have assumed that the target particle recoil is negligible, the retardation factor in the denominator of Eq. (6.14) may be set equal to unity, and we write

$$\mathbf{f}=f\exp[-i\boldsymbol{\varrho}_0\cdot\mathbf{z}(-\tau_r)],\qquad(6.14\text{a})$$

where f is just the usual free-particle scattering amplitude given by

$$f=-(2\pi)^2(p/V)Tu.$$

The quantity $(-\tau_r)$ is the time that the scattering occurred for a beam particle which reaches the point x at time zero.

In the weak binding limit, we may evaluate ψ_{sc}, Eq. (6.9), in terms of a retarded time even for a target which recoils. The evaluation is a little more involved, but straightforward.[17]

In I, our principle interest was the study of intensity correlations between beams scattered from multiparticle targets. We shall therefore discuss in a manner similar to the above treatment the scattering from a composite target containing N scatterers having coordinates $\mathbf{z}_1,\cdots,\mathbf{z}_N$ and an initial wave function $g_0(\mathbf{z}_1,\cdots,\mathbf{z}_N)$. The geometry of the scattering is illustrated in Fig. 4; the coordinate origin is located in the target. The point in the source from which the jth beam particle originated is $\mathbf{d}_j$, and the point $\mathbf{x}_j$ is at the detector. We introduce the vectors

$$\begin{aligned}\mathbf{D}_j{}^\alpha&=\mathbf{x}_j-\mathbf{z}_\alpha\,,\\ \mathbf{R}_j{}^\alpha&=\mathbf{z}_\alpha-\mathbf{d}_j\,.\end{aligned}\qquad(6.16)$$

We assume that multiple scattering may be neglected. Then, following the argument which led to the expression for the scattered wave from a single bound scatterer, Eq. (6.9), we obtain for the wave function of the jth scattered particle

$$\phi_j(\mathbf{p},\mathbf{x}_j)=-\frac{2\pi}{x(2\pi)^{3/2}}\times\sum_{\alpha=1}^{N}\int_0^\infty d\tau\int^\infty kdk\; e^{iP_j(k,\alpha,\tau)}T_j{}^\alpha u_j.\qquad(6.17)$$

[17] This is done explicitly by K. M. Watson, Phys. Rev. **118**, 886 (1960). See also *Collision Theory*, Ref. 12, Sec. (11.1).

Here u_j is the spin wave function of the jth beam particle which has momentum $\mathbf{p}$ and

$$\begin{aligned}P_j(k,\alpha,\tau)&=(i\eta+\epsilon_p-\epsilon_k)\tau\\&\quad+kD_j{}^\alpha(-\tau)+\mathbf{p}\cdot\mathbf{R}_j{}^\alpha(-\tau)\approx(i\eta+\epsilon_p-\epsilon_k)\tau\\&\quad+kx_j-\mathbf{p}\cdot\mathbf{d}_j-(k\hat{x}_j-\mathbf{p})\cdot\mathbf{z}_\alpha(-\tau)\,,\end{aligned}\qquad(6.18)$$

for $x_j\gg|\mathbf{z}_\alpha|$.

It is useful to form wave packets out of the ϕ_j according to

$$\Phi_j(\mathbf{x}_j,0)=\int d^3p\,A_j(\mathbf{p})\phi_j(\mathbf{p},\mathbf{x}_j)\,,\qquad(6.19)$$

and from these the symmetrized wave function at $t=0$ [a formal derivation of Eq. (6.20) is outlined in the Appendix]:

$$\Psi(0)=\mathcal{S}\prod_{j=1}^{n}\Phi_j(x_j,0)g_0(z_1,\cdots,z_N)\,.\qquad(6.20)$$

The time-dependent beam particle functions are of course generated by the kinetic energy operator K_j for the jth particle. Thus

$$\begin{aligned}\Phi_j(\mathbf{x}_j,t)&=e^{-iK_jt}\Phi_j(\mathbf{x}_j,0)\\&=\int d^3p\,A_j(\mathbf{p})\phi_j(\mathbf{p},\mathbf{x}_j,t).\end{aligned}\qquad(6.21)$$

For large x, and consequently large t,

$$\begin{aligned}&\phi_j(\mathbf{p},\mathbf{x}_j,t)\\&\quad\equiv e^{-iK_jt}\phi_j(\mathbf{p},\mathbf{x}_j)\\&\quad=-\frac{(2\pi)^{-1/2}}{x_j}\sum_{\alpha=1}^{N}\int_0^\infty d\tau\int kdk\,e^{iP_j(k,\alpha,\tau,t)}T_j{}^\alpha u_j,\end{aligned}\qquad(6.22)$$

where

$$\begin{aligned}P_j(k,\alpha,\tau,t)&=(i\eta+\epsilon_p-\epsilon_k)\tau\\&\quad-\epsilon_kt+kD_j{}^\alpha(-\tau)+\mathbf{p}\cdot\mathbf{R}_j{}^\alpha(-\tau)\,.\end{aligned}\qquad(6.23)$$

We write $\phi_j(p,x_j,t)$ in a form analogous to the single target expression, Eq. (6.12)

$$\phi_j(\mathbf{p},\mathbf{x}_j,t)=\frac{(2\pi)^{-3/2}}{x_j}\exp[i(px_j-\epsilon_pt)]\mathbf{f}_j\,,\qquad(6.24)$$

where the scattering amplitude $\mathbf{f}$ is given by

$$\begin{aligned}\mathbf{f}_j=-2\pi\sum_{\alpha=1}^{N}\int_0^\infty d\tau\int kdk\\ \times\exp\{i[P_j(k,\alpha,\tau,t)-px_j+\epsilon_pt]\}T_j\alpha u_j\,.\end{aligned}\qquad(6.25)$$

To evaluate the integral which appears in (6.25) we again use the weak binding, heavy scatter limit and write $k=p+q$, with q small in the exponent of (6.25)

$$\begin{aligned}P_j(k,\alpha,\tau,t)-px_j+\epsilon_p{}^x&\approx i\eta\tau+q[-V(t+\tau)\\&\quad+x_j-\hat{x}_j\cdot\mathbf{z}_\alpha(-\tau)]-\mathbf{p}\cdot\mathbf{d}_j-\boldsymbol{\varrho}_0\cdot\mathbf{z}_\alpha(-\tau)\,,\end{aligned}\qquad(6.26)$$

where, as before

$$\begin{aligned}\tau&=d\epsilon_p/d_p\,,\\ \boldsymbol{\varrho}_0&=p\hat{x}_j-p\,.\end{aligned}\qquad(6.27)$$

We find then, analogous to (6.14) and (6.15), the results

$$\phi_j(\mathbf{p},\mathbf{x},t)=\frac{(2\pi)^{-3/2}}{x}\{\exp[i(px-\epsilon_p t)]\}[\exp(-i\mathbf{p}\cdot\mathbf{d}_j)]$$
$$\times\sum_{\alpha=1}^{N} f_\alpha{}^j \exp[-i\varrho_0\cdot\mathbf{z}_\alpha(-\tau_r)],$$
$$-\tau_r=t-[x-\hat{x}\cdot\mathbf{z}_\alpha(-\tau_r)]/V \qquad (6.28)$$
$$=t-|\mathbf{x}-\mathbf{z}_\alpha(-\tau_r)|/V,$$
$$f_\alpha{}^j=-(2\pi)^2\frac{p}{V}T_j{}^\alpha u_j.$$

The wave function $\phi_j(\mathbf{p},\mathbf{x}_j,t)$ may also be written, using the definitions of $D_j{}^\alpha$ and $R_j{}^\alpha$, (6.16), as

$$\phi_j(\mathbf{p},\mathbf{x}_j,t)=\frac{(2\pi)^{-3/2}}{x}\exp(-i\epsilon_p t)$$
$$\times\sum_{\alpha=1}^{N} f_\alpha{}^j \exp\{i[pD_j{}^\alpha(-\tau_r)+\mathbf{p}\cdot\mathbf{R}_j{}^\alpha(-\tau_r)]\}. \qquad (6.29)$$

In practice we may replace $\mathbf{p}$ by $p\hat{R}_j{}^\alpha$ since, as was explained in I, those beam particles which do not strike the target are of no interest. Then the wave function ϕ_j becomes a function of the magnitude of $\mathbf{p}$ and if we define

$$a_j(p)=\int d\Omega_p p^2 A_j(\mathbf{p}),$$

we obtain

$$\Phi_j(\mathbf{x},t)=\int dp\; a_j(p)\phi_j(p,\mathbf{x},t), \qquad (6.30)$$

which is similar in form to that used in Sec. III, Eq. (3.1). [The form (6.30) also applies to a spherically expanding beam.]

The wave function for a lightly bound target particle which recoils may be obtained similarly.[17]

VII. SCATTERED PARTICLE CORRELATIONS

We may now apply the general expressions for time correlations developed in Sec. II to the analysis of a scattering experiment. In particular, we consider a modification of that illustrated in Fig. 4 shown in Fig. 5 where we add a second detector of target particles and process the output of the two detectors precisely as described in connection with Fig. 2. For simplicity of presentation here we assume that the scattering is independent of spin. The general case will be described in the second paper of this series.

The two detectors are located at $\mathbf{Y}_1$ and $\mathbf{Y}_2$. For simplicity we assume that they and the beam source are small in the sense that all three subtend very small solid angles at the target. Further, the target will be taken small enough that the retarded times defined by Eq. (6.28) are substantially the same for all target particles,

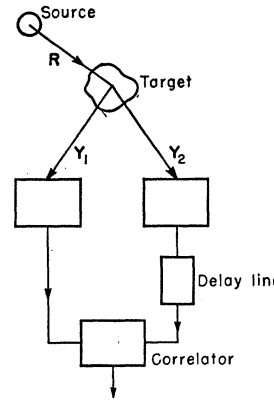

FIG. 5. Correlated counting rates for scattering by a composite target.

so that we have[18]

$$-\tau_{r_1}=t-(Y_1/V),$$
$$-\tau_{r_2}=t-(Y_2/V). \qquad (7.1)$$

We shall, finally, assume that the beam spectrum is narrow enough that the scattering amplitudes may be treated as constants over the spectral width.

It is convenient to express the instantaneous correlated counting rate defined by Eq. (2.17) in terms of the wave function at $t=0$ given by Eq. (6.20) and Heisenberg counting operators, Eq. (2.16). Thus we write Eq. (2.17) as

$$M=(\Psi(0),E_1(t_1)J_2(t_2)E_1(t_1)J_1(t_1)\Psi(0)). \qquad (7.2)$$

For the present experiment, it is desirable to arrange that M_d defined by Eq. (2.27) is negligible. Then we may apply directly Eq. (2.36), remembering that we must include now in the scalar product in (7.2) the target wave function g_0. We have, then,

$$\langle G_{12}\rangle=\bar{n}^2\int(1)\int(2)$$
$$\times\langle(g_0,[\chi(1)\chi(2)\pm g|\chi(1,2)|^2]g_0)\rangle. \qquad (7.3)$$

Here $\langle\cdots\rangle$ indicates an average over the scatterer states g_0.[19] In addition, if we were not assuming a small source, the average would also include one over the source points $\mathbf{d}_j$.[20] The mean counting rate for detector "1" becomes according to Eqs. (2.8) and (2.14a)

$$\langle G_1\rangle=B_1(0)\bar{n}\langle(g_0,\chi(1)g_0)\rangle\int_1 d^3y\gamma_1(\mathbf{y}). \qquad (7.4)$$

It is often the case, as explained in I, that one can replace Eq. (7.3) by the simpler expression

$$\langle G_{12}\rangle\cong\bar{n}^2\int(1)\int(2)[\langle(g_0,\chi(1)g_0)\rangle$$
$$\times\langle(g_0,\chi(2)g_0)\rangle\pm g|\langle(g_0,\chi(12)g_0)\rangle|^2]. \qquad (7.5)$$

[18] These simplifications are of course not essential, but permit us to avoid unnecessary detail here.

[19] If the scatterer is in thermodynamic equilibrium, for example, this will be the usual ensemble average of equilibrium statistical mechanics.

[20] This has been exhibited explictly in I.

What is involved here is that in the sum over a complete set of target states g_n which appears in a product like

$$(g_0,\chi(1)\chi(2)g_0)=\sum_n(g_0,\chi(1)g_n)(g_n,\chi(2)g_0)$$

we retain only the term $n=0$. This is usually permissible, for example, when the target is large compared to the characteristic correlation distances within it. It is well to keep in mind that (possibly) interesting higher order target correlations have been lost in the passage from the rigorous Eq. (7.3) to the approximate Eq. (7.5).

For an incoherent incident beam we may have the particular characteristics of the wave packet amplitudes described in Sec. III to express the quantities which appear in Eq. (7.5) in terms of the beam spectral function [see Eqs. (3.2) and (3.5)]:

$$\langle(g_0,\chi(1)g_0)\rangle=(2\pi)^3\frac{N_B}{\bar{n}}\int d\epsilon_p g(\epsilon_p)\times\langle(g_0,|\phi_j(p,\mathbf{y}_1,t_1)|^2g_0)\rangle,$$
$$\langle(g_0,\chi(12)g_0)\rangle=(2\pi)^3\frac{N_B}{\bar{n}}\int d\epsilon_p g(\epsilon_p)\times\langle(g_0,\phi_j^*(p,\mathbf{y}_1,t_1)\phi_j(p,\mathbf{y}_2,t_2)g_0)\rangle, \tag{7.6}$$

etc. The wave functions ϕ_j are defined by Eq. (6.28) with $\hat{p}\equiv\hat{R}_j{}^\alpha$.

Let us recall the prescription given following Eqs. (2.28) for carrying out spin orientation averages and sums. Each ϕ_j is, in general, labeled by an initial orientation for the beam and target particles. The average over the target particle orientations is obviously implied by Eq. (7.6). A corresponding average over the beam particles which are labeled by the index, j in Eq. (7.6) is also implied. Final beam spin orientations are to be specified by the detectors so that in Eqs. (7.3) and (7.5) we must sum separately over the spin orientations (which might have been observed) at detectors "1" and "2." Since the form of counting operator we have assumed is spin-independent, only the initial spin orientation is summed over in Eqs. (7.6).

To avoid complication of notation, we assume henceforth that the scattering is spin-independent and described by the wave function (6.28). Our assumption about the smallness of the source, target, and detectors permits us to write Eq. (6.28) in a somewhat simpler form:

$$\phi_j(p,\mathbf{x},t)=\frac{(2\pi)^{-3/2}}{x}\times\exp[i(px-\epsilon_p t)]\exp[ipd_j]\mathfrak{F}(\hat{x},t), \tag{7.7}$$

where

$$\mathfrak{F}(\mathbf{x},t)=\sum_{\alpha=1}^{N}f_\alpha(\hat{x})\exp[-i\boldsymbol{\varrho}\cdot\mathbf{z}_\alpha(-\tau_r)] \tag{7.8}$$

and $\boldsymbol{\varrho}=p(\hat{x}-\hat{R})$ with $\hat{R}$ a unit vector from source to target; it is the usual momentum transfer.

The first of the quantities appearing in Eq. (7.6) may now be evaluated in terms of the scattering amplitude:

$$\begin{aligned}\langle(g_0,\chi(1)g_0)\rangle&=\frac{N_B}{\bar{n}y_1{}^2}\int d\omega\, g(\omega)\langle(g_0,|\mathfrak{F}(\hat{y}_1,t)|^2g_0)\rangle\\&\cong\frac{N_B}{\bar{n}y_1{}^2}\langle(g_0,|\mathfrak{F}(\hat{y}_1,t)|^2g_0)\rangle\\&\equiv\frac{N_B}{\bar{n}y_1{}^2}\sigma(\hat{y}_1),\end{aligned} \tag{7.9}$$

where $\sigma(\hat{y}_1)$ is the differential cross section for scattering in the direction $\hat{y}_1$. The second line in (7.9) follows from our assumption about the constancy of the scattering amplitude over the beam spectrum.

The flux of scattered particles at $\mathbf{y}$ is according to Eq. (2.4)

$$\begin{aligned}F(\mathbf{y}_1)&=N_BV(\sigma(\hat{y}_1)/y_1{}^2)\\&\equiv F_T(\sigma(\hat{y}_1)/y_1{}^2),\end{aligned} \tag{7.10}$$

where the beam flux incident on the target F_T is given by

$$F_T=N_BV. \tag{7.11}$$

For the "calibrated detector" defined in Sec. II, Eqs. (2.11) and (2.14), we find from Eq. (7.4)

$$\langle G_1\rangle=B_1(0)\Sigma_1F_T\frac{\sigma(\hat{Y}_1)}{Y_1{}^2}=\Sigma_1F_T\frac{\sigma(\hat{Y}_1)}{Y_1{}^2}, \tag{7.12}$$

if $B_1(0)=1$.

The second of the expressions in (7.6) may be evaluated in terms of our wave function (7.7) to give

$$\langle(g_0,\chi(12)g_0)\rangle=\frac{N_B}{\bar{n}y_1y_2}\int d\omega g(\omega)\exp\{i[p(y_2-y_1)-\omega(t_2-t_1)]\}\langle\mathfrak{F}^*(\hat{y}_1,t_1)\mathfrak{F}(\hat{y}_2,t_2)\rangle. \tag{7.13}$$

Using our small-detector assumption, we define and write

$$\begin{aligned}\boldsymbol{\varrho}_1&\equiv p(\hat{y}_1-\hat{R})\cong p(\hat{Y}_1-\hat{R}),\\\boldsymbol{\varrho}_2&\equiv p(\hat{y}_2-\hat{R})\cong p(\hat{Y}_2-\hat{R}).\end{aligned} \tag{7.14}$$

We shall suppose that the target contains species $c=1,2,\cdots$ of particles and that for all of the α_c particles of species c, $f_{\alpha c}=f_c$. Then

$$\langle(g_0,\chi(12)g_0)\rangle=\frac{N_B}{\bar{n}y_1y_2}\sum_{c,c'}f_{c'}{}^*(\hat{y}_2)f_c(\hat{y}_1)\int d\omega g(\omega)\exp\{i[p(y_2-y_1)-\omega(t_2-t_1)]\}\times\langle(g_0,\sum_{\alpha_{c'},\alpha_c}\exp\{i[\boldsymbol{\varrho}_1\cdot\mathbf{z}_{\alpha_{c'}}(-\tau_{r1})-\boldsymbol{\varrho}_2\cdot\mathbf{z}_{\alpha_c}(-\tau_{r2})]\}g_0)\rangle, \tag{7.15}$$

where the sums on $\alpha_c(\alpha_{c'})$ run over all particles of species $c(c')$.

This result may be expressed in terms of Van Hove's correlation function[21] by introducing the particle density of species c at time t:

$$n_c(\mathbf{x},t)=\sum_{\alpha_c}\delta[\mathbf{x}-\mathbf{z}_\alpha(t)]. \tag{7.16}$$

Then if we define[22]

$$F_{c'c}(t_2,t_1)\equiv\int d^3x'\int d^3x\,\exp[i(\boldsymbol{\varrho}_1\cdot\mathbf{x}'-\boldsymbol{\varrho}_2\cdot\mathbf{x})]\langle n_{c'}[\mathbf{x}',t_1-(Y_1/V)]n_c[\mathbf{x},t_2-(Y_2/V)]\rangle, \tag{7.17}$$

we have

$$\langle(g_0,\chi(12)g_0)\rangle=\frac{N_B}{\bar{n}y_1y_2}\int d\omega\, g(\omega)\exp\{i[p(y_2-y_1)-\omega(t_2-t_1)]\}\sum_{c'c}f_{c'}^*(\hat{y}_1)f_c(\hat{y}_2)F_{c'c}(t_2,t_1). \tag{7.18}$$

For the calibrated counter, for example, we obtain from Eqs. (7.5) and (7.12)

$$\langle G_{12}\rangle=\langle G_1\rangle\langle G_2\rangle\pm\frac{F_T{}^2g}{Y_1{}^2Y_2{}^2}\int_{-\infty}^{\infty}dt_2L_2(T_2-t_2)\int_{-\infty}^{\infty}dt_1L_1(T_1-t_1)\int_1\frac{d^3y_1}{w_2}$$
$$\times\int_2\frac{d^3y_2}{w_2}\left|\int d\omega g(\omega)\exp\{i[p(y_2-y_1)-\omega(t_2-t_1)]\}\right|^2|\sum_{c'c}f_{c'}^*(\hat{Y}_1)f_c(\hat{Y}_2)F_{c'c}(t_2,t_1)|^2, \tag{7.19}$$

where the last factor has been removed from the ω integrals because of the narrow beam spectrum assumption. The autocorrelation function for a single counter may be obtained from this by supposing that the two detectors are a single one. In this case, for a detector of area Σ and rapid response time we have $\langle G_{12}\rangle=\langle G_c\rangle$,

$$\langle G_c\rangle=\langle G_1\rangle^2\pm(\Sigma/Y_1{}^2)^2F_T{}^2[B_1(0)]^2g$$
$$\times|\sum_{c'c}f_{c'}^*(\hat{Y}_1)f_c(\hat{Y}_1)F_{c'c}(T_2,T_1)|^2. \tag{7.20}$$

As has been noted earlier, a dc blocking filter may be placed in the detector outputs, causing $\langle G_1\rangle=\langle G_2\rangle=0$ in Eq. (7.19). The evaluation then follows the lines indicated in Eq. (3.10) where we imagine that the bandwidth function is very narrow in comparison to the beam spectrum.

APPENDIX: DERIVATION OF THE MANY-BEAM PARTICLE WAVE FUNCTION

To derive Eq. (6.20) from the many-body Schrödinger equation which describes all the beam particles and the target, we use the multiple-scattering form for the many-body wave function.[23] First, we define

$$d\equiv\epsilon_p+w_0+i\eta-\sum_{j=1}^n K_j-h \tag{A1}$$

[see Eqs. (6.1) and (6.21)]. In terms of d and the scattering matrices

$$\mathcal{T}_j=\sum_{\alpha=1}^n\mathcal{T}_j{}^\alpha,$$

we may write the wave function in the form

$$\Psi^+=\mathcal{S}\left\{1+\frac{1}{d}\sum_j\mathcal{T}_j+\frac{1}{d}\sum_{j_1}\mathcal{T}_{j_1}\times\frac{1}{d}\sum_{j_2(\neq j_1)}\mathcal{T}_{j_2}+\cdots\right\}\chi_a g_0. \tag{A2}$$

Here χ_a is the product of the n plane-wave functions for the n beam particles. On rearranging terms in Eq. (A2) (and assuming that no beam particle returns to scatter again on the target) we obtain

$$\Psi^+=\mathcal{S}\prod_{j=1}^n\left[1+\frac{1}{d}\mathcal{T}_j\right]\chi_a g_0. \tag{A3}$$

When the beam particle density is low enough that their mutual excitations of the target do not interfere with each other, and when appropriate wave packets are constructed, this becomes equivalent to Eq. (6.20).

[21] L. Van Hove, Phys. Rev. **95**, 249 (1954).

[22] It is often possible to replace n_c by $\delta n_c=n_c-\langle(g_0,n_cg_0)\rangle$, $[c=1,2,\cdots]$ in the expressions (7.15), since scattering from the mean densities may be negligibly small.

[23] See, for example, Eqs. (11.266) of *Collision Theory*, Ref. 12.